UNIQUENESS OF THE MAXIMAL IDEAL OF THE BANACH ALGEBRA OF BOUNDED OPERATORS ON $C([0, \omega_1])$

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Abstract. Let ω_1 be the first uncountable ordinal. By a result of Rudin, bounded operators on the Banach space $C([0, \omega_1])$ have a natural representation as $[0, \omega_1] \times [0, \omega_1]$-matrices. Loy and Willis observed that the set of operators whose final column is continuous when viewed as a scalar-valued function on $[0, \omega_1]$ defines a maximal ideal of codimension one in the Banach algebra $\mathscr{B}(C([0, \omega_1]))$ of bounded operators on $C([0, \omega_1])$. We give a coordinate-free characterization of this ideal and deduce from it that $\mathscr{B}(C([0, \omega_1]))$ contains no other maximal ideals. We then obtain a list of equivalent conditions describing the strictly smaller ideal of operators with separable range, and finally we investigate the structure of the lattice of all closed ideals of $\mathscr{B}(C([0, \omega_1]))$.

1. Introduction

Loy and Willis [17] proved that every derivation from the Banach algebra $\mathscr{B}(C([0, \omega_1]))$ of (bounded) operators on the Banach space of continuous functions on the ordinal interval $[0, \omega_1]$ equipped with its order topology into a Banach $\mathscr{B}(C([0, \omega_1]))$-bimodule is automatically continuous. At the heart of their proof is the observation that the set $\mathscr{M}$ consisting of those operators whose final column is continuous at ω_1 is a maximal ideal of codimension one in $\mathscr{B}(C([0, \omega_1]))$. We call $\mathscr{M}$ the *Loy–Willis ideal*. Its precise definition will be given in Section 2, once we have introduced the necessary terminology.

Motivated by the desire to understand the lattice of closed ideals of $\mathscr{B}(C([0, \omega_1]))$, we shall prove the following result.

Theorem 1.1. *The Loy–Willis ideal is the unique maximal ideal of $\mathscr{B}(C([0, \omega_1]))$.*

This result is in fact an immediate consequence of a more general theorem, which is of independent interest because it gives a coordinate-free characterization of the Loy–Willis ideal. (By 'coordinate-free', we mean without reference to the matrix representation of operators.)

Theorem 1.2. *An operator T on $C([0, \omega_1])$ belongs to the Loy–Willis ideal $\mathscr{M}$ if and only if the identity operator on $C([0, \omega_1])$ does not factor through T in the sense that there are no operators R and S on $C([0, \omega_1])$ such that $I = STR$.*

The implication $\Rightarrow$ is obvious because the ideal $\mathscr{M}$ is proper. The converse is much harder to prove; this will be the topic of Section 3. Once it has been proved, however,

2010 *Mathematics Subject Classification.* Primary 47L10, 46H10; Secondary 47L20, 46B26, 47B38.

Key words and phrases. Continuous functions on ordinals, bounded operators on Banach spaces, maximal ideal, Loy–Willis ideal.

Theorem 1.1 is immediate because Theorem 1.2 implies that the identity operator belongs to the ideal generated by any operator not in $\mathscr{M}$.

Many Banach spaces X share with $C([0,\omega_1])$ the property that the set

$$\mathscr{M}_X = \{T \in \mathscr{B}(X) : \text{the identity operator on } X \text{ does not factor through } T\}$$

is the unique maximal ideal of $\mathscr{B}(X)$. As noted in [5], the only non-trivial part of this statement is that $\mathscr{M}_X$ is closed under addition, and as in Theorem 1.2, this is often verified by showing that $\mathscr{M}_X$ is equal to some known ideal of $\mathscr{B}(X)$.

Banach spaces X for which $\mathscr{M}_X$ is the unique maximal ideal of $\mathscr{B}(X)$ include:

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (see [8]);

(ii) $X = L_p([0,1])$ for $1 \leqslant p < \infty$ (see [6, Theorem 1.3] and the text following it);

(iii) $X = \ell_\infty \cong L_\infty([0,1])$ (use [16, Proposition 2.f.4], as explained in [11, p. 253]);

(iv) $X = \big(\bigoplus \ell_2^n\big)_{c_0}$ and $X = \big(\bigoplus \ell_2^n\big)_{\ell_1}$ (see [12] and [13, Corollary 2.12]);

(v) $X = C([0,1])$ (use [19, Theorem 1] and [20, Theorem 1], as in [4, Example 3.5]);

(vi) $X = C([0,\omega^\omega])$ and $X = C([0,\omega^\alpha])$, where α is a countable epsilon number, that is, a countable ordinal satisfying $\alpha = \omega^\alpha$. This result is due to Philip A. H. Brooker (unpublished), who has kindly given us permission to include it here together with the following proof. Let $X = C([0,\omega^{\omega^\alpha}])$, where α is either 1 or a countable epsilon number. The set $\mathscr{S}\mathscr{Z}_\alpha(X)$ of operators on X having Szlenk index at most ω^α is an ideal of $\mathscr{B}(X)$ by [4, Theorem 2.2]. We shall discuss this ideal in more detail in Section 5; for now, it suffices to note that $\mathscr{S}\mathscr{Z}_\alpha(X) \subseteq \mathscr{M}_X$ because the identity operator on X has Szlenk index $\omega^{\alpha+1}$ (see Theorem 5.6(ii) below). Conversely, Bourgain [3, Proposition 3] has shown that each operator $T \notin \mathscr{S}\mathscr{Z}_\alpha(X)$ fixes a copy of X. Hence, using [19, Theorem 1] as above, we see that the identity operator on X factors through T, so $\mathscr{M}_X \subseteq \mathscr{S}\mathscr{Z}_\alpha(X)$, and the conclusion follows.

Note that, by [23], $C([0,\omega_1])$ differs from all of the above-mentioned Banach spaces by not being isomorphic to its Cartesian square $C([0,\omega_1]) \oplus C([0,\omega_1])$.

Having thus understood the maximal ideal(s) of $\mathscr{B}(C([0,\omega_1]))$, we turn our attention to the other closed ideals of this Banach algebra. We begin with a characterization of the ideal $\mathscr{X}(C([0,\omega_1]))$ of operators with separable range. To state it, we require three pieces of notation.

Firstly, we associate with each countable ordinal σ the multiplication operator P_σ given by $P_\sigma f = f \cdot \mathbf{1}_{[0,\sigma]}$ for $f \in C([0,\omega_1])$. Since the indicator function $\mathbf{1}_{[0,\sigma]}$ is idempotent and continuous with norm one, P_σ is a contractive projection on $C([0,\omega_1])$, and its range is isometrically isomorphic to $C([0,\sigma])$. For technical reasons (notably Theorem 1.3(a) below), we also require the rank-one perturbation

$$\widetilde{P}_\sigma = P_\sigma + \mathbf{1}_{[\sigma+1,\omega_1]} \otimes \varepsilon_{\omega_1} \tag{1.1}$$

of P_σ, where $\varepsilon_{\omega_1} \in C([0,\omega_1])^*$ denotes the point evaluation at ω_1. Clearly $\widetilde{P}_\sigma$ is a contractive projection.

Secondly, for Banach spaces X, Y and Z, we let

$$\mathscr{G}_Z(X,Y) = \operatorname{lin}\{TS : S \in \mathscr{B}(X,Z),\ T \in \mathscr{B}(Z,Y)\}. \tag{1.2}$$

This defines an operator ideal in the sense of Pietsch, the *ideal of operators factoring through* Z. Note that if Z contains a complemented copy of its square $Z \oplus Z$, then the set $\{TS : S \in \mathscr{B}(X,Z),\ T \in \mathscr{B}(Z,Y)\}$ is already closed under addition, so the 'lin' in (1.2) is superfluous. We write $\overline{\mathscr{G}}_Z(X,Y)$ for the norm closure of $\mathscr{G}_Z(X,Y)$; this is a closed operator ideal.

Thirdly, we denote by $c_0(\omega_1)$ the Banach space of scalar-valued functions f defined on $\omega_1 = [0,\omega_1)$ such that the set $\{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\}$ is finite for each $\varepsilon > 0$, equipped with the pointwise-defined vector-space operations and the supremum norm.

We can now state our characterization of the operators on $C([0,\omega_1])$ with separable range. Its proof will be given in Section 4.

Theorem 1.3. *The following five conditions are equivalent for an operator T on $C([0,\omega_1])$:*

(a) $T = \widetilde{P}_\sigma T \widetilde{P}_\sigma$ *for some countable ordinal* σ;
(b) $T \in \mathscr{G}_{C([0,\sigma])}(C([0,\omega_1]))$ *for some countable ordinal* σ;
(c) $T \in \overline{\mathscr{G}}_{C([0,\sigma])}(C([0,\omega_1]))$ *for some countable ordinal* σ;
(d) $T \in \mathscr{X}(C([0,\omega_1]))$;
(e) T *does not fix a copy of* $c_0(\omega_1)$.

Warning! Theorem 1.3 does *not* imply that the ideal $\mathscr{G}_{C([0,\sigma])}(C([0,\omega_1]))$ is closed for each countable ordinal σ, despite the equivalence of conditions (b) and (c). The reason is that, for given $T \in \overline{\mathscr{G}}_{C([0,\tau])}(C([0,\omega_1]))$ (where τ is a countable ordinal), the ordinal σ such that (b) holds may be much larger than τ and depend on T.

Finally, in Section 5, we study the entire lattice of closed ideals of $\mathscr{B}(C([0,\omega_1]))$. To classify all the closed ideals of $\mathscr{B}(C([0,\omega_1]))$ seems an impossible task. In the first instance, one would need to classify the closed ideals of $\mathscr{B}(C([0,\omega^{\omega^\alpha}]))$ for each countable ordinal α, something that already appears intractable; it has currently been achieved only in the simplest case $\alpha = 0$, where $C([0,\omega]) \cong c_0$.

Figure 1 below summarizes the findings of Section 5, using the following conventions: (i) we suppress $C([0,\omega_1])$ everywhere, thus writing $\mathscr{K}$ instead of $\mathscr{K}(C([0,\omega_1]))$ for the ideal of compact operators on $C([0,\omega_1])$, *etc.*; (ii) $\mathscr{I} \hookrightarrow \mathscr{J}$ means that the ideal $\mathscr{I}$ is properly contained in the ideal $\mathscr{J}$; (iii) a double-headed arrow indicates that there are no closed ideals between $\mathscr{I}$ and $\mathscr{J}$; (iv) α denotes a countable ordinal; and (v) $K_\alpha = [0,\omega^{\omega^\alpha}]$.

2. Preliminaries

All Banach spaces are over the scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The term *ideal* always means two-sided ideal. By an *operator*, we understand a bounded linear operator between Banach spaces. We write $\mathscr{B}(X)$ for the Banach algebra of all operators on the Banach space X, equipped with the operator norm. Since $\mathscr{B}(X)$ is unital, Krull's theorem implies that every proper ideal of $\mathscr{B}(X)$ is contained in a maximal ideal. It is well known that every non-zero ideal of $\mathscr{B}(X)$ contains the ideal $\mathscr{F}(X)$ of finite-rank operators on X.

We define the *support* of a scalar-valued function f defined on a set K by $\operatorname{supp}(f) = \{k \in K : f(k) \neq 0\}$. When K is a compact space, $C(K)$ denotes the Banach space of all

$$
\begin{tikzcd}
& \mathscr{B} & \\
& \mathscr{M} \arrow[u, hook, two heads] & \\
\mathscr{X} \arrow[r, hook] & \mathscr{X}+\overline{\mathscr{G}}_{c_0(\omega_1)} \arrow[u, hook] & \\
\vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \\
\overline{\mathscr{G}}_{C(K_{\alpha+1})} \arrow[u, hook] \arrow[r, hook] & \overline{\mathscr{G}}_{C(K_{\alpha+1})\oplus c_0(\omega_1)} \arrow[u, hook] & \\
\overline{\mathscr{G}}_{C(K_{\alpha})} \arrow[u, hook] \arrow[r, hook] & \overline{\mathscr{G}}_{C(K_{\alpha})\oplus c_0(\omega_1)} \arrow[u, hook] & \\
\vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \\
\overline{\mathscr{G}}_{C(K_1)} \arrow[u, hook] \arrow[r, hook] & \overline{\mathscr{G}}_{C(K_1)\oplus c_0(\omega_1)} \arrow[u, hook] & \overline{\mathscr{G}}_{c_0(\omega_1)} \arrow[l, hook] \\
& \overline{\mathscr{G}}_{c_0} \arrow[ul, hook] \arrow[ur, hook, two heads] & \\
& \mathscr{K} \arrow[u, hook, two heads] & \\
& \{0\} \arrow[u, hook, two heads] &
\end{tikzcd}
$$

FIGURE 1. Partial structure of the lattice of closed ideals of $\mathscr{B} = \mathscr{B}(C([0, \omega_1]))$.

continuous scalar-valued functions on K, equipped with the supremum norm. For $k \in K$, the *point evaluation* at k is the contractive functional $\varepsilon_k \in C(K)^*$ given by $\varepsilon_k(f) = f(k)$.

The *Kronecker delta* of a pair of ordinals α and β is given by $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha,\beta} = 0$ otherwise. By convention, we consider 0 a limit ordinal. For an ordinal σ, we write $[0, \sigma]$ for the set of ordinals less than or equal to σ, equipped with the order topology. This is a compact Hausdorff space which is metrizable if and only if it is separable if and only if σ is countable. (As a set, $[0, \sigma]$ is of course equal to $\sigma + 1$; we use the notation $[0, \sigma]$ to emphasize that it is a topological space.) The symbols ω and ω_1 are reserved for the first infinite and uncountable ordinal, respectively, while $\mathbb{N}$ denotes the set of positive integers. We shall use extensively the well-known fact that a scalar-valued function on $[0, \omega_1]$ is continuous at ω_1 if and only if it is eventually constant.

Suppose that σ is an infinite ordinal, and let $T \in \mathscr{B}(C([0, \sigma]))$. For each ordinal $\alpha \in [0, \sigma]$, the functional $f \mapsto Tf(\alpha)$, $C([0, \sigma]) \to \mathbb{K}$, is continuous, so by a result of

Rudin [21], there are unique scalars $T_{\alpha,\beta}$, where $\beta \in [0,\sigma]$, such that

$$\sum_{\beta\in[0,\sigma]} |T_{\alpha,\beta}| < \infty \qquad \text{and} \qquad Tf(\alpha) = \sum_{\beta\in[0,\sigma]} T_{\alpha,\beta} f(\beta) \qquad (f \in C([0,\sigma])).$$

We can therefore associate a $[0,\sigma]\times[0,\sigma]$-matrix $[T_{\alpha,\beta}]$ with T. Note that the composition ST of operators S and T on $C([0,\sigma])$ corresponds to standard matrix multiplication in the sense that

$$(ST)_{\alpha,\gamma} = \sum_{\beta\in[0,\sigma]} S_{\alpha,\beta} T_{\beta,\gamma} \qquad (\alpha,\gamma \in [0,\sigma]). \tag{2.1}$$

We shall now specialize to the case where $\sigma = \omega_1$. For $T \in \mathscr{B}(C([0,\omega_1]))$ and $\alpha \in [0,\omega_1]$, we denote by r_α^T and k_α^T the α^{th} row and column of the matrix of T, respectively, considered as scalar-valued functions defined on $[0,\omega_1]$; thus $r_\alpha^T\colon \beta \mapsto T_{\alpha,\beta}$ and $k_\alpha^T\colon \beta \mapsto T_{\beta,\alpha}$. The following result of Loy and Willis summarizes the basic properties of these functions.

Proposition 2.1. ([17, Proposition 3.1]) *Let T be an operator on $C([0,\omega_1])$. Then:*

(i) *the function r_α^T is absolutely summable for each ordinal $\alpha \in [0,\omega_1]$, hence has countable support, and*

$$\|T\| = \sup\Big\{ \sum_{\beta\in[0,\omega_1]} |T_{\alpha,\beta}| : \alpha \in [0,\omega_1] \Big\};$$

(ii) *the function k_α^T is continuous whenever $\alpha = 0$ or α is a countable successor ordinal;*

(iii) *the function k_α^T is continuous at ω_1 for each countable ordinal α;*

(iv) *the restriction of $k_{\omega_1}^T$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k_{\omega_1}^T(\alpha)$ exists.*

Note that the statement in (iv) is the best possible because the final column of the matrix associated with the identity operator is equal to $\mathbf{1}_{\{\omega_1\}}$, so it is not continuous at ω_1.

Loy and Willis studied the subspace $\mathscr{M}$ of $\mathscr{B}(C([0,\omega_1]))$ consisting of those operators T such that $k_{\omega_1}^T$ is continuous at ω_1. They observed that $\mathscr{M}$ is an ideal of codimension one, hence maximal (see [17, p. 336]); this is the *Loy–Willis ideal.* It is straightforward to verify that every operator on $C([0,\omega_1])$ not belonging to $\mathscr{M}$ has uncountably many non-zero rows and columns. Although not required here, let us mention that the key result of Loy and Willis [17, Theorem 3.5] states that the ideal $\mathscr{M}$ has a bounded right approximate identity.

3. The Loy–Willis ideal: completion of the proof of Theorem 1.2

In preparation for the proof of Theorem 1.2 ($\Leftarrow$), we require three lemmas.

Lemma 3.1. *Let T be a Fredholm operator acting on a Banach space X which is isomorphic to its hyperplanes (and hence to all its closed subspaces of finite codimension). Then the identity operator on X factors through T.*

Proof. Choose a closed subspace W of X which is complementary to $\ker T$. Then W has finite codimension in X, so W is isomorphic to X by assumption, and the restriction $\widetilde{T}\colon w \mapsto Tw$, $W \to T(X)$, is an isomorphism, hence the identity operator on X factors through $\widetilde{T}$. Now the result follows because $T(X)$ is complemented in X, so $\widetilde{T}$ factors through T. $\square$

Lemma 3.2. *Let $\Xi = (\xi_\sigma)_{\sigma\in[0,\omega_1)}$ be a strictly increasing transfinite sequence of countable ordinals, and define $\xi_{\omega_1} = \omega_1$ and $\zeta_\lambda = \sup\{\xi_\sigma : \sigma \in [0,\lambda)\}$ for each limit ordinal $\lambda \in [\omega,\omega_1]$. Then:*

(i) *the mapping U_Ξ given by $U(\mathbf{1}_{[0,\sigma]}) = \mathbf{1}_{[0,\xi_\sigma]}$ for each $\sigma \in [0,\omega_1]$ extends uniquely to a linear isometry of $C([0,\omega_1])$ onto $\overline{\operatorname{lin}}\,\{\mathbf{1}_{[0,\xi_\sigma]} : \sigma \in [0,\omega_1]\}$;*

(ii)
$$[0,\omega_1] = [0,\xi_0] \cup \bigcup_{\sigma\in[0,\omega_1)} [\xi_\sigma + 1, \xi_{\sigma+1}] \cup \bigcup_{\lambda\in[\omega,\omega_1]\ \text{limit}} [\zeta_\lambda, \xi_\lambda],$$
where the intervals on the right-hand side are pairwise disjoint;

(iii) *the mapping $\varphi_\Xi \colon [0,\omega_1] \to [0,\omega_1]$ given by*
$$\varphi_\Xi(\alpha) = \begin{cases} \xi_0 & \textit{for } \alpha \in [0,\xi_0] \\ \xi_{\sigma+1} & \textit{for } \alpha \in [\xi_\sigma + 1, \xi_{\sigma+1}],\ \textit{where } \sigma \in [0,\omega_1), \\ \zeta_\lambda & \textit{for } \alpha \in [\zeta_\lambda, \xi_\lambda],\ \textit{where } \lambda \in [\omega,\omega_1] \textit{ is a limit ordinal,} \end{cases}$$
is continuous and satisfies $\varphi_\Xi \circ \varphi_\Xi = \varphi_\Xi$; hence the associated composition operator $\Phi_\Xi \colon f \mapsto f \circ \varphi_\Xi$ defines a contractive projection of $C([0,\omega_1])$ onto the subspace $\overline{\operatorname{lin}}\,\{\mathbf{1}_{[0,\xi_\sigma]} : \sigma \in [0,\omega_1]\}$;

(iv) *the matrix associated with the operator Φ_Ξ is given by*
$$(\Phi_\Xi)_{\alpha,\beta} = \begin{cases} \delta_{\beta,\xi_0} & \textit{for } \alpha \in [0,\xi_0] \\ \delta_{\beta,\xi_{\sigma+1}} & \textit{for } \alpha \in [\xi_\sigma + 1, \xi_{\sigma+1}],\ \textit{where } \sigma \in [0,\omega_1), \\ \delta_{\beta,\zeta_\lambda} & \textit{for } \alpha \in [\zeta_\lambda, \xi_\lambda],\ \textit{where } \lambda \in [\omega,\omega_1] \textit{ is a limit ordinal.} \end{cases}$$

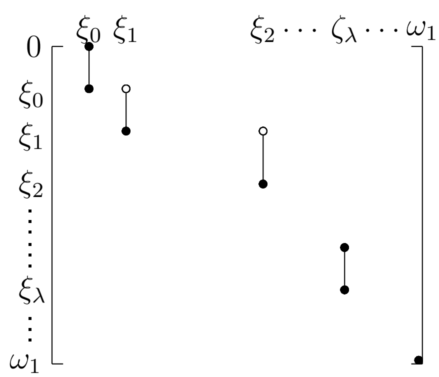

Figure 2. Structure of the the matrix associated with Φ_Ξ.

Proof. (i). For $n \in \mathbb{N}$, scalars $c_1,\ldots,c_n$ and ordinals $0 \leqslant \sigma_1 < \cdots < \sigma_n \leqslant \omega_1$, we have
$$\Big\|\sum_{j=1}^n c_j \mathbf{1}_{[0,\sigma_j]}\Big\| = \max_{1\leqslant m\leqslant n}\Big|\sum_{j=m}^n c_j\Big| = \Big\|\sum_{j=1}^n c_j \mathbf{1}_{[0,\xi_{\sigma_j}]}\Big\|.$$

Hence U_Ξ defines a linear isometry of $\operatorname{lin}\{\mathbf{1}_{[0,\sigma]} : \sigma\in[0,\omega_1]\}$ onto $\operatorname{lin}\{\mathbf{1}_{[0,\xi_\sigma]} : \sigma\in[0,\omega_1]\}$. Now the conclusion follows because the domain of U_Ξ is dense in $C([0,\omega_1])$.

(ii). This is straightforward to verify.

(iii). Clause (ii) ensures that the definition of φ_Ξ makes sense. To prove that φ_Ξ is continuous, suppose that (α_j) is a net in $[0,\omega_1]$ which converges to α. By the definition of the order topology, this means that for each $\beta<\alpha$, we have $\beta<\alpha_j\leqslant\alpha$ eventually. If $\alpha\notin\{\zeta_\lambda : \lambda\in[\omega,\omega_1]\text{ limit}\}$, then $\varphi_\Xi(\alpha_j)=\varphi_\Xi(\alpha)$ eventually, so the continuity of φ_Ξ at α is clear in this case. Otherwise $\alpha=\zeta_\lambda$ for some limit ordinal $\lambda\in[\omega,\omega_1]$, and $\varphi_\Xi(\alpha)=\zeta_\lambda$. Given $\beta<\zeta_\lambda$, we can take $\sigma\in[0,\lambda)$ such that $\beta<\xi_\sigma$. Since $\xi_\sigma<\xi_{\sigma+1}\leqslant\zeta_\lambda$, we have $\xi_\sigma<\alpha_j\leqslant\zeta_\lambda$ eventually. Hence the definition of φ_Ξ implies that $\xi_\sigma<\varphi_\Xi(\alpha_j)\leqslant\zeta_\lambda$ eventually, so that $\lim_j\varphi_\Xi(\alpha_j)=\zeta_\lambda=\varphi_\Xi(\alpha)$, as required.

We have $\varphi_\Xi\circ\varphi_\Xi=\varphi_\Xi$ because by definition $\varphi_\Xi(\alpha)$ belongs to the same interval as α in the partition of $[0,\omega_1]$ given in (ii). Hence Φ_Ξ is a contractive projection.

To determine its range, we observe that

$$\Phi_\Xi(\mathbf{1}_{[0,\alpha]})=\begin{cases}0 & \text{for } \alpha\in[0,\xi_0)\\ \mathbf{1}_{[0,\xi_\sigma]} & \text{for } \alpha\in[\xi_0,\omega_1],\ \text{where } \sigma=\sup\{\tau\in[0,\omega_1] : \xi_\tau\leqslant\alpha\}.\end{cases}\tag{3.1}$$

Consequently, we have

$$\Phi_\Xi(C([0,\omega_1]))\subseteq\overline{\operatorname{lin}}\,\{\mathbf{1}_{[0,\xi_\sigma]} : \sigma\in[0,\omega_1]\}\tag{3.2}$$

because the linear span of $\{\mathbf{1}_{[0,\alpha]} : \alpha\in[0,\omega_1]\}$ is dense in $C([0,\omega_1])$.

Conversely, (3.1) implies that $\Phi_\Xi(\mathbf{1}_{[0,\xi_\sigma]})=\mathbf{1}_{[0,\xi_\sigma]}$ for each $\sigma\in[0,\omega_1]$, so we have equality in (3.2) because Φ_Ξ has closed range.

(iv). This is clear from the definition of φ_Ξ. □

Lemma 3.3. *Let H be an uncountable subset of $[0,\omega_1]$. Then H is order-isomorphic to $[0,\omega_1]$, and the order isomorphism $\psi_H\colon[0,\omega_1]\to H$ is continuous with respect to the relative topology on H if and only if H is closed in $[0,\omega_1]$.*

Now suppose that H is closed in $[0,\omega_1]$. Then $\omega_1\in H$, and the composition operator $\Psi_H\colon f\mapsto f\circ\iota_H\circ\psi_H$, where $\iota_H\colon H\to[0,\omega_1]$ denotes the inclusion mapping, defines a contractive operator on $C([0,\omega_1])$.

Proof. Clearly H is order-isomorphic to $[0,\omega_1]$. If the order isomorphism ψ_H is continuous, then H is compact (as the continuous image of the compact space $[0,\omega_1]$) and hence closed in $[0,\omega_1]$.

Conversely, suppose that H is closed in $[0,\omega_1]$. Then ψ_H is a bijection between two compact Hausdorff spaces, so ψ_H is continuous if and only if its inverse is. Now

$$\psi_H([0,\sigma))=[0,\psi_H(\sigma))\cap H\qquad\text{and}\qquad\psi_H((\sigma,\omega_1])=(\psi_H(\sigma),\omega_1]\cap H\qquad(\sigma\in[0,\omega_1]),$$

which shows that ψ_H^{-1} is continuous because the sets $[0,\sigma)$ and $(\sigma,\omega_1]$ for $\sigma\in[0,\omega_1]$ form a subbasis for the topology of $[0,\omega_1]$.

The second part of the lemma follows immediately. □

Unlike Φ_Ξ, the matrix associated with Ψ_H cannot in general be depicted schematically; it is, however, possible in the particular case that we shall consider in the proof of Theorem 1.2, as shown in Figure 3 below.

Proof of Theorem 1.2 ($\Leftarrow$). Let $T \in \mathscr{B}(C([0,\omega_1])) \setminus \mathscr{M}$. Going through a series of reductions, we shall eventually reach the conclusion that there are operators $R, S \in \mathscr{B}(C([0,\omega_1]))$ and $F \in \mathscr{F}(C([0,\omega_1]))$ such that $STR + F = I$. Then $STR = I - F$ is a Fredholm operator, and the conclusion follows from Lemma 3.1.

We begin by reducing to the case where each column with countable index of the associated matrix vanishes eventually. Indeed, since $r^T_{\omega_1}$ is absolutely summable, we can take a countable ordinal ρ such that $T_{\omega_1,\beta} = 0$ whenever $\beta \in (\rho, \omega_1)$. Proposition 2.1(iii) then implies that k^T_β is eventually null for each $\beta \in (\rho,\omega_1)$, and hence the β^{th} column of the operator $T_1 = T(I - P_\rho)$ is eventually null for each $\beta \in [0,\omega_1)$. Note, moreover, that $T_1 \notin \mathscr{M}$ because $k^{T_1}_{\omega_1} = k^T_{\omega_1}$.

Next, perturbing T_1 by a finite-rank operator and rescaling, we can arrange that the final row and column of its matrix are equal to $\mathbf{1}_{\{\omega_1\}}$. To verify this, we observe that Proposition 2.1(iv) implies that the function $g\colon [0,\omega_1] \to \mathbb{K}$ given by

$$g(\alpha) = \begin{cases} (T_1)_{\alpha,\omega_1} & \text{for } \alpha \in [0,\omega_1) \\ \lim_{\gamma\to\omega_1}(T_1)_{\gamma,\omega_1} & \text{for } \alpha = \omega_1 \end{cases}$$

is continuous, so $G = g \otimes \varepsilon_{\omega_1}$ defines a finite-rank operator. The number $c = (T_1)_{\omega_1,\omega_1} - g(\omega_1)$ is non-zero because $T_1 \notin \mathscr{M}$, and the operator $T_2 = c^{-1}(T_1 - G)$ satisfies $k^{T_2}_\beta = c^{-1}k^{T_1}_\beta$ for each $\beta \in [0,\omega_1)$ and $k^{T_2}_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$. The latter statement implies that $T_2 \notin \mathscr{M}$, and $r^{T_2}_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$ because $k^{T_1}_\beta$ vanishes eventually for each $\beta \in [0,\omega_1)$.

We shall now inductively construct two transfinite sequences $(\eta_\sigma)_{\sigma\in[0,\omega_1)}$ and $(\xi_\sigma)_{\sigma\in[0,\omega_1)}$ of countable ordinals such that $\eta_\tau + \omega < \eta_\sigma$ and $\xi_\tau < \xi_\sigma$ whenever $\tau < \sigma$. First, let $\eta_0 = \xi_0 = 0$. Next, assuming that the sequences $(\eta_\tau)_{\tau\in[0,\sigma)}$ and $(\xi_\tau)_{\tau\in[0,\sigma)}$ have been chosen for some $\sigma \in [1,\omega_1)$, we define

$$\eta_\sigma = \begin{cases} \sup\Bigl(\{\eta_\tau + \omega\} \cup \bigcup_{\beta\in[0,\xi_\tau]} \operatorname{supp}(k^{T_2}_\beta)\Bigr) + 1 & \text{for } \sigma = \tau + 1, \text{ where } \tau \in [0,\omega_1), \\ \sup\{\eta_\tau : \tau \in [0,\sigma)\} & \text{for } \sigma \text{ a limit ordinal} \end{cases} \tag{3.3}$$

and

$$\xi_\sigma = \sup\Bigl(\{\xi_\tau + 1 : \tau \in [0,\sigma)\} \cup \bigcup_{\alpha\in[0,\eta_\sigma+\omega]} \operatorname{supp}(r^{T_2}_\alpha)\Bigr). \tag{3.4}$$

It is clear that $\xi_\tau < \xi_\sigma$ for each $\tau < \sigma$, and also that $\eta_\tau + \omega < \eta_\sigma$ if σ is a successor ordinal. On the other hand, if σ is a limit ordinal, then $\tau < \sigma$ implies that $\tau + 1 < \sigma$, so $\eta_\tau + \omega < \eta_{\tau+1} \leqslant \eta_\sigma$, as desired. Hence the induction continues.

Let $T_3 = T_2\Phi_\Xi$, where Φ_Ξ is the composition operator associated with the transfinite sequence $\Xi = (\xi_\sigma)_{\sigma\in[0,\omega_1)}$ as in Lemma 3.2(iii). Using Lemma 3.2(iv) and matrix multiplication, we see that $r^{T_3}_{\omega_1} = k^{T_3}_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$. In fact, each of the rows of the matrix of T_3 indexed

by the set $H = \bigcup_{\sigma\in[1,\omega_1)}[\eta_\sigma, \eta_\sigma+\omega]\cup\{\omega_1\}$ has (at most) one-point support. More precisely, since the sets defining H are pairwise disjoint, we can define a map $\theta\colon H\to[1,\omega_1]$ by

$$\theta(\alpha) = \begin{cases} \xi_\sigma & \text{for } \alpha\in[\eta_\sigma,\eta_\sigma+\omega],\ \text{where } \sigma\in[1,\omega_1) \text{ is a successor ordinal,}\\ \zeta_\sigma & \text{for } \alpha\in[\eta_\sigma,\eta_\sigma+\omega],\ \text{where } \sigma\in[1,\omega_1) \text{ is a limit ordinal,}\\ \omega_1 & \text{for } \alpha=\omega_1,\end{cases}$$

where $\zeta_\sigma = \sup\{\xi_\tau : \tau\in[0,\sigma)\}$ as in Lemma 3.2, and we claim that

$$\operatorname{supp}(r_\alpha^{T_3})\subseteq\{\theta(\alpha)\}\qquad(\alpha\in H). \tag{3.5}$$

This has already been verified for $\alpha=\omega_1$. Otherwise $\alpha\in[\eta_\sigma,\eta_\sigma+\omega]$ for some $\sigma\in[1,\omega_1)$, and $\omega_1\notin\operatorname{supp}(r_\alpha^{T_3})$ because $k_{\omega_1}^{T_3}=\mathbf{1}_{\{\omega_1\}}$. Given $\gamma\in[0,\omega_1)$, matrix multiplication shows that

$$(T_3)_{\alpha,\gamma} = \sum_{\beta\in[0,\omega_1]}(T_2)_{\alpha,\beta}(\Phi_\Xi)_{\beta,\gamma} = \sum_{\beta\in[0,\xi_\sigma]}(T_2)_{\alpha,\beta}(\Phi_\Xi)_{\beta,\gamma}$$

because $\alpha\leqslant\eta_\sigma+\omega$ implies that $\sup\operatorname{supp}(r_\alpha^{T_2})\leqslant\xi_\sigma$ by (3.4), so that $(T_2)_{\alpha,\beta}=0$ for $\beta\in(\xi_\sigma,\omega_1]$. Now if σ is a successor ordinal, say $\sigma=\tau+1$, then for each $\beta\in[0,\xi_\tau]$, we have $\sup\operatorname{supp}(k_\beta^{T_2})<\eta_\sigma\leqslant\alpha$ by (3.3), so that $(T_2)_{\alpha,\beta}=0$ for such β, and hence

$$(T_3)_{\alpha,\gamma} = \sum_{\beta\in[\xi_\tau+1,\xi_{\tau+1}]}(T_2)_{\alpha,\beta}(\Phi_\Xi)_{\beta,\gamma} = \begin{cases}\displaystyle\sum_{\beta\in[\xi_\tau+1,\xi_{\tau+1}]}(T_2)_{\alpha,\beta} & \text{if } \gamma=\xi_{\tau+1}=\xi_\sigma=\theta(\alpha)\\ 0 & \text{otherwise}\end{cases}$$

by Lemma 3.2(iv). Otherwise σ is a limit ordinal, and for each $\beta\in[0,\zeta_\sigma)$, we can choose $\tau\in[0,\sigma)$ such that $\beta\leqslant\xi_\tau$. Then $\sup\operatorname{supp}(k_\beta^{T_2})<\eta_{\tau+1}<\eta_\sigma\leqslant\alpha$, so that $(T_2)_{\alpha,\beta}=0$ for such β, and as above we find that

$$(T_3)_{\alpha,\gamma} = \sum_{\beta\in[\zeta_\sigma,\xi_\sigma]}(T_2)_{\alpha,\beta}(\Phi_\Xi)_{\beta,\gamma} = \begin{cases}\displaystyle\sum_{\beta\in[\zeta_\sigma,\xi_\sigma]}(T_2)_{\alpha,\beta} & \text{if } \gamma=\zeta_\sigma=\theta(\alpha)\\ 0 & \text{otherwise.}\end{cases}$$

This completes the proof of (3.5).

The set H defined above is clearly uncountable. To prove that it is also closed, let (α_j) be a net in H converging to some $\alpha\in[0,\omega_1]$. Then, for each $\beta\in[0,\alpha)$, there is j_0 such that $\beta<\alpha_j\leqslant\alpha$ whenever $j\geqslant j_0$. In particular, we may suppose that $\alpha_j\leqslant\alpha$ for each j. Let $\sigma=\sup\{\tau\in[1,\omega_1):\eta_\tau\leqslant\alpha\}\in[1,\omega_1]$. If $\sigma=\omega_1$, then $\alpha\geqslant\sup\{\eta_\tau:\tau\in[0,\omega_1)\}=\omega_1$, so that $\alpha=\omega_1\in H$. Otherwise σ is countable. The choice of σ implies that $\eta_\sigma\leqslant\alpha<\eta_{\sigma+1}$. (In the case where σ is a limit ordinal, the first inequality follows from the fact that $\eta_\sigma=\sup\{\eta_\tau:\tau\in[0,\sigma)\}$ by (3.3).) Hence, for each j, we have

$$\alpha_j\in H\cap[0,\alpha]\subseteq H\cap[0,\eta_{\sigma+1}) = \bigcup_{\tau\in[1,\sigma]}[\eta_\tau,\eta_\tau+\omega]\subseteq[0,\eta_\sigma+\omega],$$

so $\eta_\sigma+\omega\geqslant\lim_j\alpha_j=\alpha$ and thus $\alpha\in[\eta_\sigma,\eta_\sigma+\omega]\subseteq H$, as desired.

We can therefore associate with H the composition operator Ψ_H as in Lemma 3.3; Figure 3 sketches the matrix associated with Ψ_H.

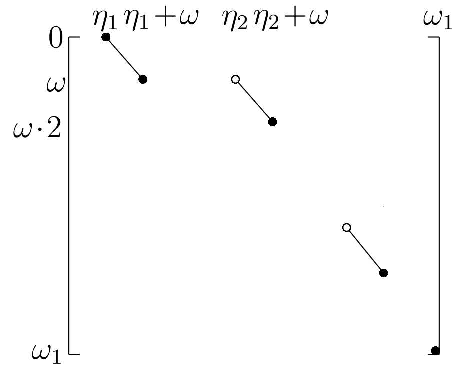

FIGURE 3. Structure of the matrix associated with Ψ_H.

Let $T_4 = \Psi_H T_3$. Then, for $f \in C([0,\omega_1])$ and $\alpha \in [0,\omega_1]$, we have

$$(T_4 f)(\alpha) = (T_3 f)(\psi_H(\alpha)) = \sum_{\beta\in[0,\omega_1]} (T_3)_{\psi_H(\alpha),\beta} f(\beta) = (T_3)_{\psi_H(\alpha),\gamma} f(\gamma) \tag{3.6}$$

by (3.5), where $\gamma = (\theta\circ\psi_H)(\alpha)$ and $\psi_H\colon [0,\omega_1] \to H$ denotes the order isomorphism as in Lemma 3.3. Taking $\alpha = \omega_1$ and $f = \mathbf{1}_{[0,\omega_1]}$, we obtain $T_4(\mathbf{1}_{[0,\omega_1]})(\omega_1) = (T_3)_{\omega_1,\omega_1} = 1$. Being continuous, the function $T_4(\mathbf{1}_{[0,\omega_1]})$ is eventually constant, so we can find a countable ordinal χ such that $T_4(\mathbf{1}_{[0,\omega_1]})(\alpha) = 1$ for each $\alpha \in [\chi,\omega_1]$. Moreover, (3.6) implies that $\operatorname{supp}(r_\alpha^{T_4}) \subseteq \{(\theta\circ\psi_H)(\alpha)\}$ for each $\alpha\in[0,\omega_1]$, hence we conclude that

$$(T_4)_{\alpha,\beta} = \delta_{(\theta\circ\psi_H)(\alpha),\beta} \qquad (\alpha\in[\chi,\omega_1],\ \beta\in[0,\omega_1]). \tag{3.7}$$

Let $T_5 = QT_4$, where $Q = I - P_\chi + \mathbf{1}_{[0,\chi]}\otimes\varepsilon_\chi$. An easy computation gives

$$Q_{\alpha,\beta} = \begin{cases} \delta_{\chi,\beta} & \text{for } \alpha\in[0,\chi] \\ \delta_{\alpha,\beta} & \text{for } \alpha\in(\chi,\omega_1] \end{cases} \qquad (\alpha,\beta\in[0,\omega_1]),$$

which together with (3.7) implies that

$$(T_5)_{\alpha,\gamma} = \sum_{\beta\in[0,\omega_1]} Q_{\alpha,\beta}(T_4)_{\beta,\gamma} = \begin{cases} \delta_{(\theta\circ\psi_H)(\chi),\gamma} & \text{for } \alpha\in[0,\chi] \\ \delta_{(\theta\circ\psi_H)(\alpha),\gamma} & \text{for } \alpha\in(\chi,\omega_1] \end{cases} \qquad (\alpha,\gamma\in[0,\omega_1]). \tag{3.8}$$

This shows in particular that $k_{\omega_1}^{T_5} = \mathbf{1}_{\{\omega_1\}}$, so $T_5\notin\mathscr{M}$, and consequently the set

$$\Gamma = \{\gamma\in[0,\omega_1] : k_\gamma^{T_5} \neq 0\} = (\theta\circ\psi_H)([\chi,\omega_1])$$

is uncountable. Let $M = (\mu_\sigma)_{\sigma\in[0,\omega_1]}$ be the increasing enumeration of Γ. We note that $\mu_0 = (\theta\circ\psi_H)(\chi)$ and $\mu_{\omega_1} = \omega_1$, and for each $\sigma\in[0,\omega_1]$, we have

$$T_5(\mathbf{1}_{[0,\mu_\sigma]}) = \mathbf{1}_{[0,\nu_\sigma]}, \qquad \text{where} \qquad \nu_\sigma = \sup\{\alpha\in[0,\omega_1] : (\theta\circ\psi_H)(\alpha)\leqslant\mu_\sigma\}. \tag{3.9}$$

The transfinite sequence $N = (\nu_\sigma)_{\sigma\in[0,\omega_1]}$ is clearly increasing; to see that it increases strictly, suppose that $0\leqslant\tau<\sigma\leqslant\omega_1$. Then $\mu_\tau<\mu_\sigma$. On the one hand, since $\mu_\sigma\in\Gamma$, we

have $\mu_\sigma = (\theta \circ \psi_H)(\alpha)$ for some $\alpha \in [\chi, \omega_1]$, and therefore

$$T_5(\mathbf{1}_{[\mu_\tau+1,\mu_\sigma]})(\alpha) = \sum_{\gamma \in [\mu_\tau+1,\mu_\sigma]} (T_5)_{\alpha,\gamma} = 1$$

by (3.8). On the other, (3.9) implies that $T_5(\mathbf{1}_{[\mu_\tau+1,\mu_\sigma]}) = \mathbf{1}_{[0,\nu_\sigma]} - \mathbf{1}_{[0,\nu_\tau]}$. The only way that this function can take the value 1 at α is if $\nu_\tau < \alpha \leqslant \nu_\sigma$, and the conclusion follows.

Lemma 3.2(i) implies that there are linear isometries U_M and U_N on $C([0,\omega_1])$ such that $U_M(\mathbf{1}_{[0,\sigma]}) = \mathbf{1}_{[0,\mu_\sigma]}$ and $U_N(\mathbf{1}_{[0,\sigma]}) = \mathbf{1}_{[0,\nu_\sigma]}$ for each $\sigma \in [0,\omega_1]$. Moreover, their ranges are complemented in $C([0,\omega_1])$ by Lemma 3.2(iii); the projection onto $U_N(C([0,\omega_1]))$ is Φ_N, and so we can define an operator $V_N = U_N^{-1}\Phi_N$ on $C([0,\omega_1])$. We now see that $V_N T_5 U_M = I$ because $V_N T_5 U_M(\mathbf{1}_{[0,\sigma]}) = \mathbf{1}_{[0,\sigma]}$ for each $\sigma \in [0,\omega_1]$, and the result follows. □

Remark 3.4. Ogden [18] extended the definition of $\mathscr{M}$ to the case of $\mathscr{B}(C([0,\omega_\eta]))$, where η is any ordinal such that ω_η is a regular cardinal. Our main result is valid also in this case with a similar proof.

4. Operators with separable range: the proof of Theorem 1.3

We require four lemmas. The first is straightforward, so we omit its proof.

Lemma 4.1. *Let K be a compact topological space, and let $(f_\alpha)_{\alpha\in[0,\omega_1)}$ be a family of pairwise disjointly supported functions in $C(K)$ such that* $\sup\{\|f_\alpha\| : \alpha \in [0,\omega_1)\} < \infty$ *and* $\inf\{\|f_\alpha\| : \alpha \in [0,\omega_1)\} > 0$. *Then $(f_\alpha)_{\alpha\in[0,\omega_1)}$ is a transfinite basic sequence equivalent to the canonical Schauder basis $(\mathbf{1}_{\{\alpha\}})_{\alpha\in[0,\omega_1)}$ for $c_0(\omega_1)$.*

Lemma 4.2. *A subspace X of $C([0,\omega_1])$ is separable if and only if X is contained in the range of the projection $\widetilde{P}_\sigma$ given by* (1.1) *for some countable ordinal σ.*

Proof. The implication $\Leftarrow$ is clear. Conversely, suppose that W is a countable dense subset of X. Since each continuous function on $[0,\omega_1]$ is eventually constant, we can choose a countable ordinal σ such that $f|_{[\sigma+1,\omega_1]}$ is constant for each $f \in W$. This implies that $\widetilde{P}_\sigma f = f$ for each $f \in W$, so as $\widetilde{P}_\sigma$ has closed range and W is dense in X, we conclude that $X \subseteq \widetilde{P}_\sigma(C([0,\omega_1]))$. □

Lemma 4.3. *Let T be an operator on $C([0,\omega_1])$ such that $T \neq \widetilde{P}_\sigma T \widetilde{P}_\sigma$ for each countable ordinal σ. Then there is an $\varepsilon > 0$ such that, for each countable ordinal ξ, there is a function $f \in C([0,\omega_1])$ with* $\operatorname{supp}(f) \subseteq (\xi,\omega_1)$ *satisfying $\|f\| \leqslant 1$ and $\|Tf\| \geqslant \varepsilon$.*

Proof by contraposition. Suppose that the conclusion is false. Then, taking $\varepsilon = 1/n$ for $n \in \mathbb{N}$, we obtain a sequence $(\xi_n)_{n\in\mathbb{N}}$ of countable ordinals such that $\|Tf\| < 1/n$ for each function $f \in C([0,\omega_1])$ with $\operatorname{supp}(f) \subseteq (\xi_n,\omega_1)$ and $\|f\| \leqslant 1$.

We claim that the countable ordinal $\xi = \sup\{\xi_n : n \in \mathbb{N}\}$ satisfies $T = T\widetilde{P}_\xi$. To verify this claim, it clearly suffices to prove that $T(I - \widetilde{P}_\xi)g = 0$ for each $g \in C([0,\omega_1])$ with $\|(I-\widetilde{P}_\xi)g\| \leqslant 1$. Letting $f = (I-\widetilde{P}_\xi)g$, we have $\operatorname{supp}(f) \subseteq (\xi,\omega_1) = \bigcap_{n\in\mathbb{N}}(\xi_n,\omega_1)$ because $P_\xi(I-\widetilde{P}_\xi) = 0$ and $f(\omega_1) = 0$. Hence the choice of ξ_n implies that $\|Tf\| < 1/n$ for each $n \in \mathbb{N}$, so $0 = Tf = T(I-\widetilde{P}_\xi)g$, and the claim follows.

In particular, T has separable range, so Lemma 4.2 implies that $T = \widetilde{P}_\eta T$ for some countable ordinal η. Since $\widetilde{P}_\alpha \widetilde{P}_\beta = \widetilde{P}_{\min\{\alpha,\beta\}}$, we conclude that $T = \widetilde{P}_\sigma T \widetilde{P}_\sigma$ is satisfied for $\sigma = \max\{\xi, \eta\}$. □

Lemma 4.4. *Let S be an operator on $C([0,\omega_1])$ with $k^S_{\omega_1} = 0$. For each pair ζ, η of countable ordinals, there is a countable ordinal $\xi \geqslant \zeta$ such that $P_\eta S(I - P_\xi) = 0$.*

Proof. Let $\xi = \sup\bigl(\{\zeta\} \cup \bigcup_{\alpha\in[0,\eta]} \operatorname{supp}(r^S_\alpha)\bigr)$. Then clearly $\zeta \leqslant \xi$, and ξ is countable because $\operatorname{supp}(r^S_\alpha)$ is countable and $S_{\alpha,\omega_1} = 0$ for each α. We show that $P_\eta S(I - P_\xi) = 0$ by verifying that $(P_\eta S(I - P_\xi))_{\alpha,\delta} = 0$ for each pair $\alpha, \delta \in [0, \omega_1]$. Indeed, by (2.1), we have

$$(P_\eta S(I-P_\xi))_{\alpha,\delta} = \sum_{\beta,\gamma\in[0,\omega_1]} (P_\eta)_{\alpha,\beta} S_{\beta,\gamma} (I - P_\xi)_{\gamma,\delta} = \begin{cases} 0 & \text{if } \alpha \in (\eta, \omega_1], \\ 0 & \text{if } \delta \in [0, \xi], \\ S_{\alpha,\delta} & \text{otherwise,} \end{cases}$$

and $S_{\alpha,\delta} = 0$ for $\alpha \in [0, \eta]$ and $\delta \in (\xi, \omega_1]$ by the choice of ξ. □

Proof of Theorem 1.3. The implications (a)⇒(b)⇒(c)⇒(d)⇒(e) are all straightforward. Indeed, (a)⇒(b) because $\widetilde{P}_\sigma$ is a rank-one perturbation of P_σ, whose range is isometrically isomorphic to $C([0,\sigma])$; (b)⇒(c) is obvious; (c)⇒(d) follows from the facts that $C([0,\sigma])$ is separable and $\mathscr{X}(C([0,\omega_1]))$ is a closed operator ideal; and (d)⇒(e) because $c_0(\omega_1)$ is non-separable.

Finally, we prove that (e)⇒(a) by contraposition. Suppose that $T \neq \widetilde{P}_\sigma T \widetilde{P}_\sigma$ for each countable ordinal σ. If $T \notin \mathscr{M}$, then Theorem 1.2 implies that T fixes a copy of $C([0,\omega_1])$ and thus of $c_0(\omega_1)$. Otherwise choose $\varepsilon > 0$ as in Lemma 4.3. By induction, we shall construct a family $(f_\alpha)_{\alpha\in[0,\omega_1)}$ of functions in $C([0,\omega_1])$ such that $\sup\{\|f_\alpha\| : \alpha \in [0,\omega_1)\} \leqslant 1$, $\inf\{\|Tf_\alpha\| : \alpha \in [0,\omega_1)\} \geqslant \varepsilon$, $f_0(\omega_1) = 0$, $Tf_0(\omega_1) = 0$ and

$$\operatorname{supp}(f_\alpha) \subseteq (\sup\operatorname{supp}(f_\beta), \omega_1) \qquad \text{and} \qquad \operatorname{supp}(Tf_\alpha) \subseteq (\sup\operatorname{supp}(Tf_\beta), \omega_1) \tag{4.1}$$

whenever $0 \leqslant \beta < \alpha < \omega_1$. Before giving the details of this construction, let us explain how it enables us to complete the proof. The families $(f_\alpha)_{\alpha\in[0,\omega_1)}$ and $(Tf_\alpha)_{\alpha\in[0,\omega_1)}$ both satisfy the conditions in Lemma 4.1, so they are equivalent to the canonical Schauder basis for $c_0(\omega_1)$. Hence, as T maps $(f_\alpha)_{\alpha\in[0,\omega_1)}$ onto $(Tf_\alpha)_{\alpha\in[0,\omega_1)}$, it fixes a copy of $c_0(\omega_1)$.

It remains to inductively construct $(f_\alpha)_{\alpha\in[0,\omega_1)}$. To start the induction, we note that $\xi = \sup(\operatorname{supp}(r^T_{\omega_1}) \setminus \{\omega_1\})$ is a countable ordinal by Proposition 2.1(i). Lemma 4.3 therefore enables us to choose a function $f_0 \in C([0,\omega_1])$ with $\operatorname{supp}(f_0) \subseteq (\xi, \omega_1)$ such that $\|f_0\| \leqslant 1$ and $\|Tf_0\| \geqslant \varepsilon$. Of the conditions that f_0 must satisfy, only $Tf_0(\omega_1) = 0$ is not evident; however, we have

$$Tf_0(\omega_1) = \sum_{\beta\in[0,\omega_1]} T_{\omega_1,\beta} f_0(\beta) = 0 \tag{4.2}$$

because $f_0(\beta) = 0$ for $\beta \in [0,\xi] \cup \{\omega_1\}$, while $T_{\omega_1,\beta} = 0$ for $\beta \in (\xi, \omega_1)$ by the choice of ξ.

Now let $\alpha \in (0, \omega_1)$, and assume inductively that functions $(f_\beta)_{\beta\in[0,\alpha)}$ in $C([0,\omega_1])$ have been chosen as specified. The function $k^T_{\omega_1}$ is continuous because $T \in \mathscr{M}$, so we may define

a rank-one operator by $F = k_{\omega_1}^T \otimes \varepsilon_{\omega_1}$. Since $k_{\omega_1}^{T-F} = 0$, we can apply Lemma 4.4 with

$$\zeta = \sup\biggl((\operatorname{supp}(r_{\omega_1}^T) \setminus \{\omega_1\}) \cup \bigcup_{\beta\in[0,\alpha)} \operatorname{supp}(f_\beta)\biggr) \qquad \text{and} \qquad \eta = \sup\biggl(\bigcup_{\beta\in[0,\alpha)} \operatorname{supp}(Tf_\beta)\biggr)$$

to obtain a countable ordinal $\xi \geqslant \zeta$ such that $P_\eta(T-F)(I-P_\xi) = 0$. (Note that the ordinals ζ and η are countable because f_β and Tf_β are continuous functions on $[0,\omega_1]$ mapping ω_1 to 0, so they have countable supports for each $\beta \in [0,\alpha)$.) By Lemma 4.3, we can take a function $f_\alpha \in C([0,\omega_1])$ with $\operatorname{supp}(f_\alpha) \subseteq (\xi,\omega_1)$ such that $\|f_\alpha\| \leqslant 1$ and $\|Tf_\alpha\| \geqslant \varepsilon$. It remains to check that (4.1) holds for each $\beta \in [0,\alpha)$. The first statement is clear because $\operatorname{supp}(f_\alpha) \subseteq (\xi,\omega_1)$ and $\sup\operatorname{supp}(f_\beta) \leqslant \zeta \leqslant \xi$. To verify the second, we observe that $Tf_\alpha(\omega_1) = 0$ by an argument similar to that given in (4.2) above. Moreover, since $f_\alpha \in \ker P_\xi$ and $f_\alpha \in \ker \varepsilon_{\omega_1} = \ker F$, we have

$$P_\eta T f_\alpha = P_\eta(T-F)(I-P_\xi)f_\alpha = 0.$$

Consequently, $\operatorname{supp}(Tf_\alpha) \subseteq (\eta,\omega_1)$, from which the desired conclusion follows because $\sup\operatorname{supp}(Tf_\beta) \leqslant \eta$. Hence the induction continues. $\square$

5. The lattice of closed ideals of $\mathscr{B}(C([0,\omega_1]))$

The aim of this section is to establish the hierarchy among the closed ideals of $\mathscr{B}(C([0,\omega_1]))$ shown in Figure 1. Beginning from the bottom of the diagram, we note that as $C([0,\omega_1])$ is a $\mathscr{L}_\infty$-space, it has the bounded approximation property, so $\mathscr{K}(C([0,\omega_1]))$ is the closure of the ideal of finite-rank operators and thus the minimum non-zero closed ideal.

To prove the minimality of the next two inclusions in Figure 1, we require the following variant of Sobczyk's theorem for $C([0,\omega_1])$, which is due to Argyros *et al.*

Proposition 5.1 ([2, Proposition 3.2])**.** *Let X be a subspace of $C([0,\omega_1])$ which is isomorphic to either c_0 or $c_0(\omega_1)$. Then X is automatically complemented.*

Remark 5.2. The first part of Proposition 5.1 follows easily from our results and Sobczyk's theorem. Indeed, let X be a subspace of $C([0,\omega_1])$ which is isomorphic to c_0. Then X is separable, hence contained in $\widetilde{P}_\sigma(C([0,\omega_1]))$ for some countable ordinal σ by Lemma 4.2. Sobczyk's theorem implies that X is complemented in $\widetilde{P}_\sigma(C([0,\omega_1]))$, and as $\widetilde{P}_\sigma(C([0,\omega_1]))$ is complemented in $C([0,\omega_1])$, so is X.

Proposition 5.3. *The identity operator on c_0 factors through each non-compact operator on $C([0,\omega_1])$. Hence no closed ideal of $\mathscr{B}(C([0,\omega_1]))$ lies strictly between $\mathscr{K}(C([0,\omega_1]))$ and $\overline{\mathscr{G}}_{c_0}(C([0,\omega_1]))$.*

Proof. This is a standard argument which we outline for completeness. Since $[0,\omega_1]$ is scattered, $C([0,\omega_1])^* \cong \ell_1([0,\omega_1])$, so $C([0,\omega_1])^*$ has the Schur property. Hence all weakly compact operators on $C([0,\omega_1])^*$ are compact. The theorems of Gantmacher and Schauder then imply that all weakly compact operators on $C([0,\omega_1])$ are compact, and therefore, by a theorem of Pełczyński, each non-compact operator on $C([0,\omega_1])$ fixes a copy of c_0. Now the conclusion follows from Proposition 5.1. $\square$

For each countable ordinal α, let Q_α denote the α^{th} projection associated with the canonical Schauder basis $(\mathbf{1}_{\{\beta\}})_{\beta\in[0,\omega_1)}$ for $c_0(\omega_1)$; that is, $(Q_\alpha f)(\beta) = f(\beta)$ for $\beta \in [0,\alpha]$ and $(Q_\alpha f)(\beta) = 0$ for $\beta \in (\alpha,\omega_1)$. We can use the projections Q_α to characterize the separable subspaces of $c_0(\omega_1)$ in a similar fashion to Lemma 4.2 for $C([0,\omega_1])$. Although this characterization follows easily from standard results such as [10, Proposition 5.6], we outline a short, elementary proof.

Lemma 5.4. *A subspace X of $c_0(\omega_1)$ is separable if and only if X is contained in the range of the projection Q_α for some countable ordinal α.*

Proof. The implication $\Leftarrow$ is immediate because Q_α has separable range for each $\alpha \in [0,\omega_1)$.

Conversely, suppose that X is separable, and let W be a dense, countable subset of X. Since each element of $c_0(\omega_1)$ has countable support, the ordinal $\alpha = \sup\bigcup_{f\in W}\operatorname{supp} f$ is countable, and clearly $Q_\alpha f = f$ for each $f \in W$. Hence W is contained in the range of Q_α, which is closed, so the same is true for X. □

Proposition 5.5. *No closed ideal of $\mathscr{B}(C([0,\omega_1]))$ lies strictly between $\overline{\mathscr{G}}_{c_0}(C([0,\omega_1]))$ and $\overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$.*

Proof. Given $T \in \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$, we consider two cases. If $T \in \mathscr{X}(C([0,\omega_1]))$, then Theorem 1.3 shows that $T = \widetilde{P}_\sigma T\widetilde{P}_\sigma$ for some countable ordinal σ. Given $\varepsilon > 0$, choose operators $U\colon C([0,\omega_1]) \to c_0(\omega_1)$ and $V\colon c_0(\omega_1) \to C([0,\omega_1])$ such that $\|T - VU\| \leqslant \varepsilon$. Since the range of the operator $U\widetilde{P}_\sigma$ is separable, we can take a countable ordinal α such that $Q_\alpha U\widetilde{P}_\sigma = U\widetilde{P}_\sigma$ by Lemma 5.4. Hence we have

$$\|T - VQ_\alpha U\widetilde{P}_\sigma\| = \|T\widetilde{P}_\sigma - VU\widetilde{P}_\sigma\| \leqslant \|T - VU\|\,\|\widetilde{P}_\sigma\| \leqslant \varepsilon,$$

so $T \in \overline{\mathscr{G}}_{c_0}(C([0,\omega_1]))$ because $Q_\alpha \in \mathscr{G}_{c_0}(c_0(\omega_1))$.

Otherwise $T \notin \mathscr{X}(C([0,\omega_1]))$, and Theorem 1.3 implies that T fixes a copy X of $c_0(\omega_1)$. Proposition 5.1 ensures that $T(X)$ is complemented in $C([0,\omega_1])$, so the closed ideal generated by T is equal to $\overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$. □

To complete Figure 1, we require the *Szlenk index* as it enables us to distinguish the $C(K)$-spaces considered therein. This ordinal-valued index, denoted by $\operatorname{Sz} X$, was originally introduced by Szlenk [24] for Banach spaces X with separable dual and has subsequently been generalized to encompass all Asplund spaces (or all Banach spaces, provided that one is willing to accept that $\operatorname{Sz} X$ takes the value 'undefined' (or ∞) if X is not an Asplund space). We shall not state the definition of the Szlenk index here as all we need to know is its value for certain $C(K)$-spaces. The interested reader is referred to [10, Section 2.4] for a modern introduction to the Szlenk index.

A proof of the first part of the following theorem is outlined in [7, Exercise 8.55], while the second, much deeper, part is due to Samuel [22]; a simplified proof of it, due to Hájek and Lancien, can be found in [9] or [10, Theorem 2.59].

Theorem 5.6. (i) *The Szlenk index of $c_0(\omega_1)$ is ω.*
(ii) *Let α be a countable ordinal. Then $C([0,\omega^{\omega^\alpha}])$ has Szlenk index $\omega^{\alpha+1}$.*

In fact, a Szlenk index can be associated with each operator between Banach spaces in such a way that the Szlenk index of a Banach space is equal to that of its identity operator. We are interested in this notion because Brooker [4, Theorem 2.2] has shown that, for each ordinal α, the collection $\mathscr{S}\mathscr{Z}_\alpha$ of operators having Szlenk index at most ω^α forms a closed operator ideal in the sense of Pietsch.

Armed with this information, we can prove that all inclusions are proper in each of the two infinite ascending chains in Figure 1.

Proposition 5.7. *Let α be a countable ordinal, and let $K_\alpha = [0,\omega^{\omega^\alpha}]$. Then:*

(i) $\overline{\mathscr{G}}_{C(K_\alpha)}(C([0,\omega_1])) \subsetneq \overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)}(C([0,\omega_1]))$;
(ii) $\overline{\mathscr{G}}_{C(K_\alpha)}(C([0,\omega_1])) \subsetneq \overline{\mathscr{G}}_{C(K_{\alpha+1})}(C([0,\omega_1]))$;
(iii) $\overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)}(C([0,\omega_1])) \subsetneq \overline{\mathscr{G}}_{C(K_{\alpha+1})\oplus c_0(\omega_1)}(C([0,\omega_1]))$;
(iv) $\overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1])) \subsetneq \overline{\mathscr{G}}_{C(K_1)\oplus c_0(\omega_1)}(C([0,\omega_1]))$.

Proof. To prove (i), let Q be a projection on $C([0,\omega_1])$ whose range is isomorphic to $c_0(\omega_1)$. Then $Q \in \overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)}(C([0,\omega_1]))$, but $Q \notin \overline{\mathscr{G}}_{C(K_\alpha)}(C([0,\omega_1]))$ because its range is non-separable.

We shall prove (ii) and (iii) simultaneously by displaying an operator belonging to $\overline{\mathscr{G}}_{C(K_{\alpha+1})}(C([0,\omega_1])) \setminus \overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)}(C([0,\omega_1]))$. More precisely, we claim that the projection P_σ is such an operator for $\sigma = \omega^{\omega^{\alpha+1}}$. Indeed, $P_\sigma \in \overline{\mathscr{G}}_{C(K_{\alpha+1})}(C([0,\omega_1]))$ because its range is isometrically isomorphic to $C(K_{\alpha+1})$. On the other hand, Theorem 5.6(ii) implies that the identity operator on $C(K_{\alpha+1})$ does not belong to the operator ideal $\mathscr{S}\mathscr{Z}_{\alpha+1}$. Since it factors through P_σ, we deduce that $P_\sigma \notin \mathscr{S}\mathscr{Z}_{\alpha+1}(C([0,\omega_1]))$, and consequently we have $P_\sigma \notin \overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)}(C([0,\omega_1]))$ because $\overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)} \subseteq \mathscr{S}\mathscr{Z}_{\alpha+1}$ as the following calculation shows

$$\operatorname{Sz} C(K_\alpha)\oplus c_0(\omega_1) = \max\{\operatorname{Sz} C(K_\alpha), \operatorname{Sz} c_0(\omega_1)\} = \omega^{\alpha+1}.$$

Here, the first equality follows from [4, Proposition 1.5(v)] (which in turn is a consequence of [9, Equation (2.3)]) and the second from Theorem 5.6.

Finally, (iv) follows by taking $\alpha = 0$ in (iii) because $C(K_0) = C([0,\omega]) \cong c_0$, so that $C(K_0)\oplus c_0(\omega_1) \cong c_0(\omega_1)$. $\square$

We now come to the most interesting result in this section.

Theorem 5.8. *The ideal $\mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$ is closed, and*

$$\mathscr{X}(C([0,\omega_1])) \subsetneq \mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1])) \subsetneq \mathscr{M}.$$

Proof. To show that the ideal $\mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$ is closed, let T be an operator belonging to its closure, and take sequences $(R_n)_{n\in\mathbb{N}}$ in $\mathscr{X}(C([0,\omega_1]))$ and $(S_n)_{n\in\mathbb{N}}$ in $\overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$ such that $R_n + S_n \to T$ as $n \to \omega$. Then $\operatorname{lin}\bigcup_{n\in\mathbb{N}} R_n(C([0,\omega_1]))$ is a separable subspace of $C([0,\omega_1])$, so Lemma 4.2 implies that it is contained in the range of $\widetilde{P}_\sigma$ for some countable ordinal σ. Hence we have $\widetilde{P}_\sigma R_n = R_n$ for each $n \in \mathbb{N}$, and therefore

$$(I-\widetilde{P}_\sigma)S_n = (I-\widetilde{P}_\sigma)(R_n+S_n) \to (I-\widetilde{P}_\sigma)T \qquad \text{as} \qquad n\to\omega,$$

so $(I - \widetilde{P}_\sigma)T \in \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$. Since $\widetilde{P}_\sigma$ has separable range, we conclude that

$$T = \widetilde{P}_\sigma T + (I - \widetilde{P}_\sigma)T \in \mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1])),$$

as required.

We have $\mathscr{X}(C([0,\omega_1])) \subsetneq \mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$ because $C([0,\omega_1])$ contains a complemented subspace isomorphic to $c_0(\omega_1)$, which is non-separable.

The proof that $\mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$ is properly contained in the Loy–Willis ideal $\mathscr{M}$ is somewhat more involved. By Theorem 1.1, it suffices to display an operator belonging to the latter, but not the former ideal. We construct such an operator by considering an operator whose range is contained in a certain $C(K)$-subspace of $C([0,\omega_1])$.

Let $H = \{\omega^\lambda : \lambda \in [\omega,\omega_1] \text{ is a limit ordinal}\}$; note that $\omega_1 \in H$ because $\omega^{\omega_1} = \omega_1$. We define an equivalence relation $\sim$ on $[0,\omega_1]$ by

$$\alpha \sim \beta \qquad \Longleftrightarrow \qquad (\alpha = \beta \quad \text{or} \quad \alpha,\beta \in H) \qquad\qquad (\alpha,\beta \in [0,\omega_1]).$$

Denote by K the quotient space $[0,\omega_1]/\!\sim$ equipped with the quotient topology, and let $\pi\colon [0,\omega_1] \to K$ be the quotient map. Then K is compact (as the continuous image of a compact space), and the composition operator $C_\pi\colon f \mapsto f \circ \pi$, $C(K) \to C([0,\omega_1])$, is a linear isometry.

For each $\alpha \in [0,\omega_1] \setminus H$, either $\alpha \in [0,\omega^\omega)$ or $\alpha \in (\omega^\lambda, \omega^{\lambda+\omega})$ for some limit ordinal $\lambda \in [\omega,\omega_1)$. In the first case, let $A_\alpha = [0,\alpha]$, in the second, let $A_\alpha = (\omega^\lambda, \alpha]$. Then A_α is clopen in $[0,\omega_1]$ and disjoint from H, so $\pi(A_\alpha)$ is clopen in K.

This implies that K is Hausdorff. Indeed, given two distinct points $\pi(\alpha), \pi(\beta) \in K$, we may suppose that $\alpha \notin H$ and $\alpha < \beta$. Then $\pi(A_\alpha)$ and $K \setminus \pi(A_\alpha)$ are disjoint open neighbourhoods of $\pi(\alpha)$ and $\pi(\beta)$, respectively.

Moreover, we can define a linear map $U\colon \operatorname{lin}\{\mathbf{1}_{[0,\alpha]} : \alpha \in [0,\omega_1]\} \to C(K)$ by

$$U\mathbf{1}_{[0,\alpha]} = \begin{cases} \mathbf{1}_{\pi(A_\alpha)} & \text{for } \alpha \in [0,\omega_1] \setminus H \\ 0 & \text{for } \alpha \in H \end{cases} \tag{5.1}$$

because $\pi(A_\alpha)$ being clopen ensures that the indicator function $\mathbf{1}_{\pi(A_\alpha)}$ is continuous. To prove that U is bounded, we consider the action of U on a function of the form $f = \sum_{j=1}^n c_j \mathbf{1}_{[0,\alpha_j]}$, where $n \in \mathbb{N}$, $c_1,\ldots,c_n \in \mathbb{K}$ and $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_n \leqslant \omega_1$. We have $\|f\| = \max\bigl\{|\sum_{j=m}^n c_j| : 1 \leqslant m \leqslant n\bigr\}$, while for $\beta \in [0,\omega_1]$,

$$(Uf)(\pi(\beta)) = \sum_{j\in J} c_j \mathbf{1}_{\pi(A_{\alpha_j})}(\pi(\beta)) = \sum_{j\in J} c_j \mathbf{1}_{A_{\alpha_j}}(\beta),$$

where $J = \{j \in \{1,\ldots,n\} : \alpha_j \notin H\}$ and the second equality follows because A_{α_j} is disjoint from H for each $j \in J$. Thus $(Uf)(\pi(\beta)) = 0$ if $\beta \notin \bigcup_{j\in J} A_{\alpha_j}$. Now suppose that $\beta \in A_{\alpha_j}$ for some $j \in J$. If $\beta \in [0,\omega^\omega)$, we let $\lambda = 0$, and otherwise we choose a limit ordinal $\lambda \in [\omega,\omega_1)$ such that $\beta \in (\omega^\lambda, \omega^{\lambda+\omega})$. Then, letting

$$k = \min\{j \in \{1,\ldots,n\} : \beta \leqslant \alpha_j\} \qquad \text{and} \qquad m = \max\{j \in \{1,\ldots,n\} : \alpha_j < \omega^{\lambda+\omega}\},$$

we have $\beta \in A_{\alpha_j}$ if and only if $k \leqslant j \leqslant m$, so

$$(Uf)(\pi(\beta)) = \sum_{j=k}^{m} c_j = \sum_{j=k}^{n} c_j - \sum_{j=m+1}^{n} c_j,$$

and consequently $|(Uf)(\pi(\beta))| \leqslant \left|\sum_{j=k}^{n} c_j\right| + \left|\sum_{j=m+1}^{n} c_j\right| \leqslant 2\|f\|$. This proves that U is bounded with norm at most two. (In fact $\|U\| = 2$ because $f = -2\mathbf{1}_{\{0\}} + \mathbf{1}_{[0,\omega^\omega]} \in C([0,\omega_1])$ has norm one, so $\|U\| \geqslant \|Uf\| = 2\|\mathbf{1}_{\{\pi(0)\}}\| = 2$.)

Since the subspace $\operatorname{lin}\{\mathbf{1}_{[0,\alpha]} : \alpha \in [0,\omega_1]\}$ is dense in $C([0,\omega_1])$, U extends uniquely to an operator of norm two defined on $C([0,\omega_1])$. We now claim that the operator $V = C_\pi U$ belongs to $\mathscr{M}\setminus(\mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1])))$. Once verified, this claim will complete the proof.

We have $V \in \mathscr{M}$ because $k^V_{\omega_1} = 0$. Indeed, given $\alpha \in [0,\omega_1]$, we shall prove that $V_{\alpha,\omega_1} = 0$ by direct computation. Since r^V_α has countable support, we can choose a non-zero countable limit ordinal λ such that $V_{\alpha,\beta} = 0$ for each $\beta \in (\omega^\lambda, \omega_1)$. Then

$$V_{\alpha,\omega_1} = \sum_{\beta \in (\omega^\lambda,\omega_1]} V_{\alpha,\beta} = (V\mathbf{1}_{(\omega^\lambda,\omega_1]})(\alpha) = C_\pi(U\mathbf{1}_{[0,\omega_1]} - U\mathbf{1}_{[0,\omega^\lambda]})(\alpha) = 0,$$

where the final equality follows from (5.1) because ω_1 and ω^λ both belong to H.

To show that $V \notin \mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$, assume the contrary, say $V = R + S$, where $R \in \mathscr{X}(C([0,\omega_1]))$ and $S \in \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1]))$. By Theorem 1.3, we can choose a countable ordinal σ such that $R = \widetilde{P}_\sigma R \widetilde{P}_\sigma$, and thus

$$(I - \widetilde{P}_\sigma)V = (I - \widetilde{P}_\sigma)S \in \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1])). \tag{5.2}$$

Take a non-zero countable ordinal τ such that $\sigma \leqslant \omega^{\omega^\tau}$, and let $\lambda = \omega^\tau$. Further, let $\iota\colon (\omega^\lambda, \omega^\lambda\cdot 2] \to [0,\omega_1]$ be the inclusion map, and define $\rho\colon [0,\omega_1] \to (\omega^\lambda, \omega^\lambda \cdot 2]$ by

$$\rho(\alpha) = \begin{cases} \omega^\lambda + 1 & \text{for } \alpha \in [0,\omega^\lambda] \\ \alpha & \text{for } \alpha \in (\omega^\lambda, \omega^\lambda\cdot 2) \\ \omega^\lambda \cdot 2 & \text{for } \alpha \in [\omega^\lambda \cdot 2, \omega_1]. \end{cases}$$

Clearly ρ is continuous, and we claim that the diagram

$$\begin{tikzcd} C((\omega^\lambda,\omega^\lambda\cdot 2]) \arrow[rrr, "I"] \arrow[d, "C_\rho"'] & & & C((\omega^\lambda,\omega^\lambda\cdot 2]) \\ C([0,\omega_1]) \arrow[r, "P_{\omega^\lambda\cdot 2}"] & C([0,\omega_1]) \arrow[r, "V"] & C([0,\omega_1]) \arrow[r, "I-\widetilde{P}_\sigma"] & C([0,\omega_1]) \arrow[u, "C_\iota"'] \end{tikzcd} \tag{5.3}$$

is commutative, where $C_\rho\colon f \mapsto f\circ\rho$ and $C_\iota\colon f \mapsto f \circ \iota$ denote the composition operators associated with ρ and ι, respectively. To verify this claim, it suffices to check the action on each function of the form $\mathbf{1}_{(\omega^\lambda,\alpha]}$, where $\alpha \in (\omega^\lambda, \omega^\lambda\cdot 2]$, because such functions span

a dense subspace of $C((\omega^\lambda, \omega^\lambda \cdot 2])$. We have $C_\rho \mathbf{1}_{(\omega^\lambda,\alpha]} = \mathbf{1}_{[0,\alpha]}$ for $\alpha \in (\omega^\lambda, \omega^\lambda \cdot 2)$ and $C_\rho \mathbf{1}_{(\omega^\lambda,\omega^\lambda\cdot 2]} = \mathbf{1}_{[0,\omega_1]}$, so $P_{\omega^\lambda\cdot 2} C_\rho \mathbf{1}_{(\omega^\lambda,\alpha]} = \mathbf{1}_{[0,\alpha]}$ for each $\alpha \in (\omega^\lambda, \omega^\lambda \cdot 2]$. Hence, by (5.1),

$$C_\iota(I - \widetilde{P}_\sigma) V P_{\omega^\lambda\cdot 2} C_\rho \mathbf{1}_{(\omega^\lambda,\alpha]} = C_\iota(I - \widetilde{P}_\sigma) C_\pi \mathbf{1}_{\pi(A_\alpha)} = C_\iota(I - \widetilde{P}_\sigma)\mathbf{1}_{A_\alpha} = \mathbf{1}_{A_\alpha},$$

which proves the claim because $A_\alpha = (\omega^\lambda, \alpha]$.

The map $\alpha \mapsto \omega^\lambda + 1 + \alpha$, $[0, \omega^\lambda] \to (\omega^\lambda, \omega^\lambda \cdot 2]$, is a homeomorphism, so the Banach spaces $C([0, \omega^\lambda])$ and $C((\omega^\lambda, \omega^\lambda \cdot 2])$ are isometrically isomorphic. Hence $C((\omega^\lambda, \omega^\lambda \cdot 2])$ has Szlenk index $\omega^{\tau+1}$ by Theorem 5.6(ii).

On the other hand, Theorem 5.6(i) implies that $\overline{\mathscr{G}}_{c_0(\omega_1)} \subseteq \mathscr{S}\mathscr{Z}_1$, so by (5.2)–(5.3) (the identity operator on) $C((\omega^\lambda, \omega^\lambda \cdot 2])$ has Szlenk index at most ω, contradicting the conclusion of the previous paragraph. □

Remark 5.9. We shall here outline an alternative, more abstract, approach to part of the proof of Theorem 5.8 given above as it sheds further light on a construction therein and raises an interesting question at the end. Our starting point is the observation that the compact Hausdorff space K defined in the proof of Theorem 5.8 is in fact just a convenient realization of the one-point compactification of the disjoint union of the intervals $[0, \omega^{\omega^\alpha}]$ for $\alpha \in [0, \omega_1)$.

A space is *Eberlein compact* if it is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some index set Γ. Being compact metric spaces, the intervals $[0, \omega^{\omega^\alpha}]$ are Eberlein compact whenever α is countable. Therefore, by a result of Lindenstrauss [14, Proposition 3.1], the one-point compactification of their disjoint union is Eberlein compact; that is, our space K is Eberlein compact. On the other hand, the interval $[0, \omega_1]$ is not Eberlein compact.

A Banach space X is *weakly compactly generated* if it contains a weakly compact subset W such that $X = \overline{\operatorname{lin}}\, W$. Amir and Lindenstrauss [1] have shown that a compact space L is Eberlein compact if and only if the Banach space $C(L)$ is weakly compactly generated. Hence, returning to our case, we see that $C(K)$ is weakly compactly generated, whereas $C([0, \omega_1])$ is not. This implies that the closed ideal $\overline{\mathscr{G}}_{C(K)}(C([0, \omega_1]))$ is proper and thus contained in the Loy–Willis ideal $\mathscr{M}$. By definition, the operator V defined in the proof of Theorem 5.8 factors through $C(K)$. On the other hand, we showed there that it does not belong to $\mathscr{X}(C([0, \omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0, \omega_1]))$, so this ideal is distinct from $\overline{\mathscr{G}}_{C(K)}(C([0, \omega_1]))$.

To prove that $\mathscr{X}(C([0, \omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0, \omega_1]))$ is contained in $\overline{\mathscr{G}}_{C(K)}(C([0, \omega_1]))$, consider first an ordinal λ of the form ω^τ, where $\tau \in [1, \omega_1)$. Replacing $\widetilde{P}_\sigma$ with 0 in (5.3), we obtain a commutative diagram as before, and since V factors through $C(K)$ and $C([0, \omega^\lambda]) \cong C((\omega^\lambda, \omega^\lambda \cdot 2])$, we conclude that the identity operator on $C([0, \omega^\lambda])$ factors through $C(K)$. Hence $C(K)$ contains a complemented copy of $C([0, \omega^\lambda])$, and therefore we have $\mathscr{X}(C([0, \omega_1])) \subseteq \mathscr{G}_{C(K)}(C([0, \omega_1]))$ by Theorem 1.3. Secondly, Lemma 4.1 implies that $(\mathbf{1}_{\{\pi(\omega^\lambda+1)\}})$, where λ ranges over all non-zero countable limit ordinals, is a transfinite basic sequence in $C(K)$ equivalent to the canonical Schauder basis for $c_0(\omega_1)$. Proposition 5.1 ensures that the closed linear span of this sequence is complemented in $C([0, \omega_1])$ and hence also in the subspace $C(K)$, so $\mathscr{G}_{c_0(\omega_1)} \subseteq \mathscr{G}_{C(K)}$.

Thus, to summarize, we have shown that

$$\mathscr{X}(C([0,\omega_1])) + \overline{\mathscr{G}}_{c_0(\omega_1)}(C([0,\omega_1])) \subsetneq \overline{\mathscr{G}}_{C(K)}(C([0,\omega_1])) \subseteq \mathscr{M}.$$

We do not know whether the final inclusion is proper; we conjecture that it is.

Another interesting question is whether the inclusion

$$\overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)}(C([0,\omega_1])) \subseteq \mathscr{S}\mathscr{Z}_{\alpha+1}(C([0,\omega_1])),$$

established in the proof of Proposition 5.7, is proper for some, or each, countable ordinal α.

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